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LETTER TO THE EDITOR

Diagonal interface in the two-dimensional Ising ferromagnet

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Abstract. The surface tension of a diagonal interface for the two-dimensional Ising ferromagnet is determined; it is direction dependent, but the critical exponent is always unity. The associated interface is diffuse.

Consider the two-dimensional Ising ferromagnet on a lattice Λ: at each vertex $\boldsymbol{i} = (i_1, i_2)$, with $-M \leqslant i_1 \leqslant M$, $-N \leqslant j_2 \leqslant N$, there is a spin $\sigma_{\boldsymbol{i}} = \pm 1$. The energy of a configuration of spins $\{\sigma\}$ is

$$E_{\Lambda,\mathscr{B}}(\{\sigma\}) = -J \sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} \sigma_{\boldsymbol{i}}\sigma_{\boldsymbol{j}} + \mathscr{B}_\Lambda(\{\sigma\}) \tag{1}$$

where $\mathscr{B}_\Lambda(\{\sigma\})$ is a boundary term (Gallavotti 1972a, b).

Let $\partial\Lambda_\pm$ denote the top and bottom and let $\partial\Lambda_{\mathrm{l,r}}$ denote the left and right edges of Λ. Then $\mathscr{B}_\Lambda^{+-}(t)$ is defined by:

$\sigma_{\boldsymbol{i}} = +1$ if $\boldsymbol{i} \in \partial\Lambda_{\mathrm{r}}$; if $\boldsymbol{i} \in \partial\Lambda_+$, and $t \leqslant i_1 \leqslant M$; if $\boldsymbol{i} \in \partial\Lambda_-$ and $-t \leqslant i_1 \leqslant -M$

$\sigma_{\boldsymbol{i}} = -1$ if $\boldsymbol{i} \in \partial\Lambda_{\mathrm{l}}$, elsewhere on $\partial\Lambda_+ \cup \partial\Lambda_-$.

The special case $t = 0$ has been treated at length elsewhere (Abraham and Reed 1974, 1975).

The canonical probability measure at temperature T for a configuration $\{\sigma\}$ is:

$$p_{\Lambda,\mathscr{B}}(\{\sigma\}) = Z(\Lambda, \mathscr{B})^{-1} \exp(-\beta E_{\Lambda,\mathscr{B}}(\{\sigma\})) \tag{2}$$

with $\beta = 1/k_{\mathrm{B}}T$ and $Z(\Lambda, \beta)$ being the partition function.

The surface tension is:

$$\tau = -\lim_{N\to\infty} [2(N+|t|)]^{-1} \lim_{M\to\infty} \ln(Z(\Lambda, \mathscr{B}^{+-}(t))/Z(\Lambda, \mathscr{B}^+)) \tag{3}$$

where $\mathscr{B}^+$ has $\sigma_{\boldsymbol{i}} = +1$ for all $\boldsymbol{i} \in \partial\Lambda$. Evidently, we have taken the incremental free energy per unit Euclidean length on $\mathbb{Z}^2$. The relationship between alternative surface tension definitions for this model has been discussed elsewhere (Fisher and Ferdinand 1967, Camp and Fisher 1972, Onsager 1944, Abraham *et al* 1973).

With the parametrisation $t = N \tan\theta$, $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, $\tau(\theta)$ has been obtained exactly:

$$\tau(\theta) = (\gamma(\omega_0) + \mathrm{i}\omega_0 \tan\theta)/(1 + |\tan\theta|) \tag{4}$$

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where (Onsager 1944)

$$\cosh \gamma(\omega) = \cosh 2K \coth 2K - \cos \omega \tag{5}$$

with $\gamma(\omega) > 0$ for real ω, $K = \beta J$ and

$$(\partial_\omega \gamma)(\omega_0) = -\mathrm{i} \tan \theta. \tag{6}$$

The function $\tau(\theta)$ has the following properties:

(i) $$\tau(\theta) = \tau(-\theta) = \tau(\tfrac{1}{2}\pi - \theta) \tag{7}$$

from the lattice symmetry.

(ii) $$\tau(0) = 2K - \ln \coth K. \tag{8}$$

This is the celebrated Onsager formula (Onsager 1944, see also Abraham *et al* 1973).

(iii) $$0 \leq \tau(\tfrac{1}{4}\pi) \leq \tau(\theta) \leq \tau(0) \tag{9}$$

and $(\partial_\theta \tau)(\frac{1}{4}\pi) = 0$. Thus $\frac{1}{4}\pi$ gives the minimum value.

(iv) The critical exponent $\mu(\theta)$ is defined as:

$$\mu(\theta) = \lim_{T \to T_c^-} \ln \tau(\theta) / \ln(1 - T/T_c) \tag{10}$$

where T_c is given by:

$$\sinh 2K_c = 1. \tag{11}$$

Using (8) and (9) and $\tau(\frac{1}{4}\pi) \sim \tau(0)/\sqrt{2}$ as $T \to T_c^-$ we have

$$\mu(\theta) = 1 \qquad \text{for all } \theta \tag{12}$$

but the amplitude is θ-dependent.

(v) An interface of given Euclidean length has a resistance to being bent; equivalently, there is a 'corner' tension, this is:

$$(\partial_\theta \tau)(\theta) = (\mathrm{i}\omega_0 - \gamma(\omega_0))/(1 + \sin 2\theta). \tag{13}$$

Note that $(\partial_\theta \tau)(\theta) > 0$ on $[0, \frac{1}{4}\pi)$, and in fact

$$(\partial_\theta \tau)(0) = -\tau(0) \tag{14}$$

which is a curious result.

The associated interface profile is defined as

$$F(p|N) = \lim_{M \to \infty} \sum_{\{\sigma\}} p_{\Lambda, \mathscr{B}^{+-}(t)}(\{\sigma\}) \sigma_{(x,y)} \tag{14}$$

with $x = p/(1 + |\tan \theta|)$ and $y = p \tan \theta / (1 + |\tan \theta|)$. When $p = \alpha[N(1 + |\tan \theta|)]^\delta$, $\delta \geq 0$ we have

$$\lim_{N \to \infty} F(p|N) = \begin{cases} 0 & 0 \leq \delta < \frac{1}{2} \\ m^* \operatorname{sgn} \alpha & \delta > \frac{1}{2} \end{cases} \tag{15}$$

where

$$m^* = [1 - (\sinh 2k)^{-4}]^{1/8} \tag{16}$$

is the spontaneous magnetisation (Onsager 1949, Yang 1952, Benettin *et al* 1973, Gallavotti 1972a, b).

When $\delta=\frac{1}{2}$ we have (see method in Abraham and Reed 1975):

$$\lim_{N\to\infty} F(\alpha N^{1/2}|N) = m^* \operatorname{sgn}\alpha\ \Phi((\gamma^{(2)}(\omega_0))^{-1/2}|\alpha|)$$

where

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,du. \tag{17}$$

The apparent connection with Gaussian random variables is curious. Two factors are noteworthy: the critical value of δ is always $\frac{1}{2}$ and the profile on this scale is analytic except at $T=0$ and $T=T_c$.

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